INSTITUTE OF MATHEMATICAL ECONOMICS

Working Paper No. 350

(fasc. 14, “Procedural Approaches to Conflict Resolution”)

Trading Bargaining Weights

Ulrike Ervig

Institute of Mathematical Economics
Bielefeld University
P.O. Box 100131
33501 Bielefeld
Germany
uervig@wiwi.uni-bielefeld.de

Claus-Jochen Haake

Institute of Mathematical Economics
Bielefeld University
P.O. Box 100131
33501 Bielefeld
Germany
chaake@wiwi.uni-bielefeld.de

October 2003 (revised version)

Abstract

We consider a model, in which two agents are engaged in two separate bargaining problems. We introduce a notion of bargaining weights (bargaining power), which is basically given by asymmetric versions of the Perles-Maschler bargaining solution. Thereby, we view bargaining power as ordinary goods that can be traded in an exchange economy. With equal initial endowment of bargaining power there exists a Walrasian equilibrium in this exchange economy. The utility allocation in equilibrium coincides with the Perles-Maschler bargaining solution of the aggregate bargaining problem. Equilibrium prices are given by the standard traveling times of the two bargaining problems (see Perles-Maschler (1981)).

Keywords: Bargaining Power, Perles-Maschler Solution, Equilibrium Model
JEL Classification: C78, C62, D51, D63

We would like to thank Joachim Rosenmüller and Walter Trockel for stimulating discussions and especially an anonymous referee for many helpful comments on an earlier version. The paper was originated while the authors were fellows of the research group “Procedural Approaches to Conflict Resolution” at the Center for Interdisciplinary Research (ZiF), Bielefeld. We thank Paul W. Thurner for suggesting the problem discussed in this paper. Haake acknowledges financial support from the project “Fairness und Anreize”, FiF-Projekt, Bielefeld University. Ervig acknowledges financial support from the project “Procedural Approaches to Conflict Resolution”, ZiF Bielefeld.

1 Introduction

The main idea in the paper evolved during various discussions about how negotiations between two different countries concerning separate issues take place and how solutions are reached. Examples are subsidies for the agricultural sector or expenditures on environmental projects. Not only delegates of a country are familiar with a frequently observed principle: "You do me a favor here and I do you a favor there". But does this method of exchanging favors lead to a "desirable" solution? Our paper clearly answers this question with yes. There are two main properties the final solution should satisfy. First, it should be Pareto optimal in the aggregate, i.e. there is no other *package* of subsidies and expenditures that makes both countries better off. Second, if one compares the final solution with the scenario, in which both issues are treated separately, then neither of the countries should be worse off in the final solution. So the favor exchange really should do a favor to both.

The latter property is known in economic theory as *superadditivity*. Perles & Maschler (1981b) introduced a superadditive bargaining solution for two person bargaining games. They discuss the symmetric version of the solution and briefly point out how asymmetric versions look like. Switching from the symmetric version to one asymmetric one can be viewed as benefitting some party. In view of this we can interpret "doing a favor" as agreeing with application of a non-symmetric bargaining solution that favors the other. In the paper we will introduce the term *bargaining weight* to quantify these benefits. Then "doing a favor" means shifting bargaining weight to the opponent.

Let us describe the idea of the paper more detailed now. The bargaining theoretic approach to finding an efficient superadditive solution is to apply the Perles-Maschler (PM) solution concept to the aggregate bargaining problem. In this version agents *agree* on the solution concept and hence, on the final solution. In the present paper we follow a different approach. We setup an exchange economy in which bargaining weights are traded. We show that there always exists an equilibrium so that the equilibrium allocation (of weights) is transformed to the PM solution of the aggregate bargaining problem. Transformation means that application of asymmetric PM solutions (with equilibrium weights) yields the symmetric PM solution in the aggregate. In particular, we do not have to compute the aggregate bargaining problem at all in order to determine the final solution. Equilibrium prices are given by primitives of the bargaining problems themselves. Hence, in our approach the agents reach the same final solution by *individual utility maximization*. It turns out that superadditivity is the key property to achieve this result. No other bargaining solution can be “decentralized“ in this way.

There are few references in the literature concerning superadditive solutions in the bargaining context. Definitely, one reason for this is that the superadditivity axiom (together with the “usual” axioms) is incompatible with the presence of more than two players. A counterexample is given in Perles (1982). However, Calvo & Gutherrez (1994) extend the construction of the PM solution to n-person bargaining games, but their solution of course loses the superadditivity property.

The organization of the paper is as follows: Section 2 provides the bargaining theoretic framework and reviews the definition of the Perles-Maschler solution. In Section 3 the basic model is introduced and discussed. Section 4 discusses the main results of the paper on existence and uniqueness of equilibria in the exchange economy and the resulting utility allocations in the aggregate bargaining problem. Examples are given in Section 5. Section 6 concludes.

2 Basic Definitions and Notation

An (axiomatic) **bargaining problem** for two persons is a pair $V := (U, \underline{x})$ consisting of a closed and convex set $U \subseteq \mathbb{R}^2$ describing feasible allocations of utilities and a vector $\underline{x} \in U$ that reflects the agents' utilities, when no agreement can be reached. Throughout the paper we will make the following assumption:

Assumption 1
For each bargaining problem $V = (U, \underline{x})$ the set U is comprehensive (i.e. $x \in U$ and $y \leq x$ implies $y \in U$). The set of individual rational allocations $U_{\underline{x}} := \{u \in U \,|\, u \geq \underline{x}\}$ is bounded (hence compact). Moreover, each U is generated by its individual rational utility allocations, i.e. $U = compH(U_{\underline{x}})$, where $compH(\cdot)$ denotes the comprehensive hull operator.

Let $\mathcal{U}^c$ denote the class of bargaining problems that satisfy Assumption 1 and denote by $\mathcal{U}_0^c$ the subclass in $\mathcal{U}^c$ that consists of bargaining problems having the common disagreement point $\underline{x} = 0$. W.l.o.g. we will restrict our analysis on the class $\mathcal{U}_0^c$. There we can and will identify V with U.

A mapping $\varphi : \mathcal{U}_0^c \longrightarrow \mathbb{R}^2$ is said to satisfy the symmetry axiom (SYM), if $\pi(\varphi(U)) = \varphi(\pi(U))$ is satisfied for each $U \in \mathcal{U}_0^c$ where $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the function that “changes

coordinates", i.e. $\pi(x_1, x_2) := (x_2, x_1)$[1]. Such a mapping φ is said to be covariant with (affine) linear transformations of utility (COV), if for each $U \in \mathcal{U}_0^c$ and each (affine) linear function $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form $L(x) = (\rho_1 x_1, \rho_2 x_2) + d$ ($\rho \in \mathbb{R}^2_{++}, d \in \mathbb{R}^2$) the condition $\varphi(L(U)) = L(\varphi(U))$ holds. Since the class $\mathcal{U}_0^c$ is not invariant under affine transformations, we restrict transformations to be linear with positive coefficients. But the analysis in the paper is valid on $\mathcal{U}^c$ and affine transformations.
A **bargaining solution (b.s.)** on $\mathcal{U}_0^c$ is a mapping $\varphi : \mathcal{U}_0^c \longrightarrow \mathbb{R}^2$ such that for each $U \in \mathcal{U}_0^c$ the solution $\varphi(U)$ is feasible (FEAS), individually rational (IR) and Pareto optimal (PO) in U. Moreover, φ has to satisfy SYM and COV.

A bargaining solution φ on $\mathcal{U}_0^c$ is said to be **superadditive** (SUPA), if it satisfies $\varphi(U^1) + \varphi(U^2) \leq \varphi(U^1 + U^2)$ for any $U^1, U^2 \in \mathcal{U}_0^c$.

It is well known that there is one and only one continuous[2] bargaining solution $\mu : \mathcal{U}^c \to \mathbb{R}^2$ that satisfies FEAS, PO, IR, COV, SYM and SUPA. This solution was introduced in Perles & Maschler (1981b). We briefly review its construction.[3]

We make use the following notation in the description of a bargaining problem. Fix $U \in \mathcal{U}_0^c$. Let $\tau_i(U) = \tau_i := \max\{t \mid t\, e^i \in U\}$ denote the maximal possible utility for agent i among individual rational utility allocations.[4] By ∂U we denote the Pareto boundary of $U_{\underline{x}}$. W.l.o.g. we will also assume that ∂U does not contain line segments parallel to the axes. With such restrictions, we can describe ∂U as the graph of a function $C : [0, \tau_1] \longrightarrow \mathbb{R}$ with $C(t) := \max\{z \in \mathbb{R} \mid (t, z) \in U\}$. Due to the convexity assumption for bargaining problems the function C is continuous, strictly decreasing, concave and it is differentiable at all but at most countably many points $t \in [0, \tau_1]$. For this reason, we may use $C'(\cdot)$ to denote the first derivative of C, taking into account that this is almost everywhere well-defined.

A **parametrization** of U is a continuous mapping $x : [a, b] \longrightarrow \partial U$ with $a, b \in \mathbb{R}, a \leq b$ such that $x(a) = (0, \tau_2), x(b) = (\tau_1, 0)$ and $x_1(\cdot)$ is non-decreasing (which implies $x_2(\cdot)$ is non-increasing).
The mapping C itself generates a canonic[5] parametrization $x^C : [0, \tau_1] \longrightarrow \mathbb{R}^2$ with

[1] Here π applied to a bargaining problem in $\mathcal{U}^c$ yields a bargaining problem in $\mathcal{U}^c$ with exchanged roles of the agents.

[2] i.e. continuous with respect to the Hausdorff topology on $\mathcal{U}_0^c$

[3] see also Perles & Maschler (1981a), Peters (1992) or Rosenmüller (2000)

[4] e^i denotes the i-th unit vector in $\mathbb{R}^2$.

[5] One can equivalently describe ∂U by a concave function $D : [0, \tau_2] \longrightarrow \mathbb{R}^2$ and derive a corresponding parametrization x^D. The term "canonic" reflects a "natural" choice of the domain of the parametrization.

$x^C(t) := (t, C(t))$. For the canonic parametrization[6], we define a function $f : [0, \tau_1] \longrightarrow \mathbb{R}$,

(1) $$f(t) := \int_0^t \sqrt{-\dot{x}_1^C(s) \cdot \dot{x}_2^C(s)}\; ds = \int_0^t \sqrt{-C'(s)}\; ds$$

(where $\dot{x}_i^C(\cdot)$ denotes the derivative of $x_i^C(\cdot)$).

The **Perles-Maschler bargaining solution** μ on $\mathcal{U}_0^c$ (hereafter PM solution) is now determined as follows: First, compute the real number $\bar{T} = \bar{T}(U)$ that satisfies

(2) $$\int_0^{\bar{T}} \sqrt{-C'(s)}\; ds = \int_{\bar{T}}^{\tau_1} \sqrt{-C'(s)}\; ds \qquad \text{i.e. } f(\bar{T}) = \frac{f(\tau_1)}{2}.$$

Second, the PM solution is defined by $\mu(U) := x^C(\bar{T}(U))$. In fact, μ is well defined as it does not depend on the parametrization used in (1) and (2) (for details see Rosenmüller (2000) or Perles & Maschler (1981b)).

The function f is continuous and strictly increasing, because the integrand is strictly positive except for $s = 0$ (here $C'(0)$ might be zero). Denote by $\bar{b}(U) = \bar{b} := f(\tau_1)$ the largest possible value that f attains. One can show that $\bar{b}$ does not depend on the parametrization chosen in (1). Hence, f is a bijection from the interval $[0, \tau_1]$ onto the interval $[0, \bar{b}]$. By $h := f^{-1}$ we denote its inverse mapping. With the mapping h and the canonic parametrization x^C, we get a new parametrization $\xi : [0, \bar{b}] \longrightarrow \mathbb{R}^2$ with $\xi := x^C \circ h$. In fact, the mapping $h : [0, \bar{b}] \longrightarrow [0, \tau_1]$ describes the transformation of parameters when changing the parametrization from x^C to ξ.

A straightforward computation now yields

(3) $$\begin{aligned} \dot{\xi}_1(s) \cdot \dot{\xi}_2(s) &= \dot{x}_1^C(h(s)) \cdot h'(s) \cdot \dot{x}_2^C(h(s)) \cdot h'(s) \\ &= C'(h(s)) \cdot (h'(s))^2 \\ &= C'(h(s)) \cdot \left(\frac{1}{f'(h(s))}\right)^2 = -1 \end{aligned}$$

Hence, computing the PM solution with the parametrization ξ, we get

$$\bar{T} = \int_0^{\bar{T}} \sqrt{-\dot{\xi}_1(s) \cdot \dot{\xi}_2(s)}\; ds = \int_{\bar{T}}^{\bar{b}} \sqrt{-\dot{\xi}_1(s) \cdot \dot{\xi}_2(s)}\; ds = \bar{b} - \bar{T}$$

and therefore obtain $\mu(U) = \xi(\bar{b}/2)$.

[6] In fact, we can take any parametrization.

Let us pause for an interpretation. As Perles & Maschler (1981b) argue one can view the PM solution as follows. There are two particles moving along the Pareto frontier. We will associate each particle with one player. Player 1's particle starts at $(0, \tau_2)$ whereas player 2's particle starts at $(\tau_1, 0)$. The interval $[0, \bar{b}]$ reflects time. They "move" on the boundary according to the parametrization ξ, i.e. the product of coordinate velocities equals -1. In view of this, we detect $\bar{b}(U)$ as the time needed to traverse the whole boundary. We therefore call $\bar{b}$ the *standard traveling time*. Hence, after time $s \in [0, \bar{b}]$ player 1's particle is located at $\xi(s)$, whereas player 2's particle stands at $\xi(\bar{b} - s)$. At time $\bar{b}/2$ the two particles meet at the PM solution.

3 A Model for Bargaining Weights Exchange

In this section we will discuss the basic model. Suppose there are two agents being engaged in two (different) bargaining problems U^A and $U^B \in \mathcal{U}_0^c$. An agreement consists of a pair $(u^A, u^B) \in U^A \times U^B$ specifying a utility allocation for each of the bargaining problems separately. We assume that utility scales are chosen in a way that does not only allow interpersonal utility comparison but also enables us to compute an agent's total utility by adding his utilities in U^A and U^B. This additivity constraint is in fact a hard condition to satisfy in certain contexts. It is necessary that there are no correlations between U^A and U^B. Therefore, an agent may evaluate an agreement by aggregating ("adding up") his valuations of u^A and u^B. In view of the covariance axiom for bargaining solutions, which enables us to (linearly) rescale the axes, this condition loses a lot of its restrictive face.

Alternatively, one can think of a model with uncertainty. There are two bargaining problems from which one will be chosen at random. In order to employ expected utility theory, one has to be able to compare utilities across bargaining problems. Then the superadditivity axiom guarantees that for both agents the expected bargaining solution w.r.t U^A and U^B is not greater than the bargaining solution of the expected bargaining problem.

The main problem, however, is that in general an agreement is not efficient w.r.t. the aggregate bargaining problem, which is given by the sum $U = U^A + U^B$. A first (naive) approach from bargaining theory would be the following. One computes the sum of the bargaining problems, and applies some bargaining solution φ to U which automatically determines some agreement (u^A, u^B) that fulfills $u^A + u^B = \varphi(U^A + U^B)$. Of course, this final agreement should be compared with the utility allocations $\varphi(U^A)$ and $\varphi(U^B)$, respectively. In case that φ is not superadditive, then in the final agreement one of the agents could be worse off in both bargaining problems compared to what φ dictates.

Yet, does this "procedure" to achieve efficiency reflects what happens in real conflicts? We often observe that agents start with an efficient focal point in each of the bargaining problems (e.g. they start with $\mu(U^A)$ and $\mu(U^B)$) and then deviate from this by favoring one agent in situation A and the other in situation B. The idea of our model is to engage a "Walrasian mechanism" to ensure efficiency. For this we construct an exchange economy, in which, roughly speaking, bargaining power is traded and initial endowments are determined by the PM solution in A and B, respectively.

We keep the notation from the previous section and attach superscripts A and B to distinguish the quantities in the referring bargaining problems. With the standard parametrizations ξ^A, ξ^B we could interpret the quantities $\bar{b}^A$ and $\bar{b}^B$ as the time a particle needs to move from $(0, \tau_2^A)$ to $(\tau_1^A, 0)$ (or the other way round), when the law of motion is determined by (3). Starting with the PM solution corresponds to letting each agent's particle move half of the standard traveling time in each problem. We will now let agents trade fractions of these traveling times. For this, consider functions $w^A, w^B : [0,1] \longrightarrow \mathbb{R}^2$ with

$$\begin{aligned} w_1^A(\alpha) &:= \xi_1^A(\alpha \cdot \bar{b}^A) = h^A(\alpha \cdot \bar{b}^A) \qquad & w_2^A(\alpha) &:= \xi_2^A((1-\alpha) \cdot \bar{b}^A) = C^A(h^A((1-\alpha) \cdot \bar{b}^A)) \\ w_1^B(\beta) &:= \xi_1^B(\beta \cdot \bar{b}^B) = h^B(\beta \cdot \bar{b}^B) \qquad & w_2^B(\beta) &:= \xi_2^B((1-\beta) \cdot \bar{b}^B) = C^B(h^B((1-\beta) \cdot \bar{b}^B)) \end{aligned}$$

For example, the quantity $w_1^A(\alpha)$ denotes the utility that agent 1 obtains in A, if he were allowed to "move his particle" from $(0, \tau_2(U^A))$ according to ξ^A for $\alpha \cdot \bar{b}^A$ (units of time). Analogously, $w_2^A(\alpha)$ reflects agent 2's utility, if he were allowed to travel $\alpha \cdot \bar{b}^A$ units of time. By straightforward computations, the point on ∂U^A that agent 2's particle reaches is exactly the point, where agent 1's particle would be after $(1-\alpha) \cdot \bar{b}^A$ time units.

Lemma 1
The functions $w_i^K (K = A, B\,,\, i = 1,2)$ are strictly increasing and concave. If $C^K (K = A, B)$ is strictly concave then so is $w_i^K (i = 1,2)$.

Proof:
We drop superscripts A, B for simplification and assume first that the function C is twice continuously differentiable. We compute derivatives of ξ_1 and ξ_2:

$$(4) \qquad {\xi_1}'(s) = h'(s) = \frac{1}{f'(h(s))} = \frac{1}{\sqrt{-C'(h(s))}} > 0 \qquad (s \in (0, \bar{b}))$$

$$(5) \qquad {\xi_1}''(s) = h''(s) = \frac{-f''(h(s)) \cdot (h)'(s)}{(f'(h(s)))^2} = \frac{C''(h(s))}{2\,(\sqrt{-C'(h(s))})^4} \le 0. \;\; (s \in (0, \bar{b}))$$

(use derivatives of the function f and observe that $C'(t) < 0$ for $t > 0$). This means that ξ_1 is strictly increasing (first derivative in (4) may not be defined for $h(s) = 0$) and concave. In view of (5) the function ξ_1 is strictly concave, if and only if C is strictly concave.

For $\xi_2 = C \circ h$ we obtain by using (4) and (5):

(6) $\quad {\xi_2}'(s) = C'(h(s)) \cdot h'(s) < 0 \quad (s \in (0, \bar{b}))$

(7) $\quad {\xi_2}''(s) = C''(h(s)) \cdot ((h)'(s))^2 + C'(h(s)) \cdot (h)''(s) = -\dfrac{C''(h(s))}{2 \cdot C'(h(s))} \leq 0 \quad (s \in (0, \bar{b}))$

Hence, ξ_2 is strictly decreasing and concave. As above, it is strictly concave if and only if C is strictly concave. Since the mappings w_1 and w_2 are compositions of ξ_1 and ξ_2 with linear transformations of the unit interval, they are also (strictly) concave. w_1 and w_2 are strictly increasing as the derivative of the linear function $\alpha \mapsto (1 - \alpha) \cdot \bar{b}$ has negative derivative.
A re-inspection of (4) and (5) reveals that the differentiability assumptions are in fact not needed. The first derivative of ξ_1 exists at all but at most countably many points in $[0, \bar{b}]$. With (strict) monotonicity of C' we get (strict) monotonicity of h', which implies the desired concavity property. $\square$

Two remarks are in order:
1) It may appear slightly dubious to express the multi-faceted notion of *bargaining power* by a simple parameter $\alpha \in [0, 1]$. However, we do not want to characterize *bargaining power* itself, but to describe the effects of "exerting bargaining power α". And these effects should be described by the utility allocation resulting in a specific bargaining problem. Hence, we can formally describe the effects of bargaining power by a mapping $P : \mathcal{U}_0^c \times [0, 1] \longrightarrow \mathbb{R}^2$ that assigns to each bargaining problem U and each bargaining weight α (of agent 1) a utility allocation $P(U, \alpha)$. Generally, there are two kinds of plausible properties that P should satisfy. First, conditions for a fixed bargaining problem and varying weight, and second, conditions for fixed weight and varying bargaining problems. Thereby, we think of the following conditions: For fixed bargaining problem the mapping $P_1(U, \cdot)$ should be strictly increasing (i.e. a gain of power should always pay off), normalized (i.e. $P_1(U, 0) = 0$, "no power yields no utility") and concave (the additional gain of utility from an additional small unit of power should decrease with the amount of power the agent already possesses). For fixed weight α we want to require the "usual" regularity conditions, such as covariance with (affine) linear transformations and in addition superadditivity. This means in effect we require $P(\cdot, \alpha)$ to be an (asymmetric) bargaining

solution.

Lemma 1 shows that all these natural conditions are satisfied by our formal notion of bargaining weight. Set for example $P(U^A, \alpha) = \xi^A(\alpha\, \bar{b}^A) = (w_1^A(\alpha), w_2^A(1-\alpha))$. Conversely, Perles & Maschler (1981a, Thm 5.1) state that any weight function exhibiting these properties (monotonicity, concavity, superadditivity) is essentially of the above described form. Therefore, we could as well start with these properties for weight functions and end up with our construction. In this spirit, we view this as a justification to speak of a parameter α to represent (agent 1's) bargaining weight in U^A.

2) Perles & Maschler (1981b) provide an economic interpretation of the "law of motion", according to which the two particles move along the Pareto boundary (see also Calvo & Gutherrez (1994)). Their idea can be described in the present context roughly as follows. We look at a fixed distribution of weights, say $(\alpha, 1-\alpha)$ and consider the ratio $\frac{w_1'(\alpha)}{w_1'(1-\alpha)}$. Linearizing first derivatives, this is roughly $\frac{w_1(\alpha+\varepsilon)-w_1(\alpha)}{w_1(1-\alpha-\varepsilon)-w_1(1-\alpha)}$ for small $\varepsilon > 0$. Thus, the numerator is agent 1's utility gain from an extra ε of power, whereas the denominator reflects his utility loss, when having weight $1-\alpha$ and losing ε. Hence the denominator describes his utility loss, when agent 2's bargaining weight were α and he gets an extra ε. Then the law of motion incorporated in eq. (3) requires such ratios of utility gain and loss to be equal, i.e.

$$\frac{\text{utility gain for 1}}{\text{utility loss for 1}} \simeq \frac{w_1'(\alpha)}{w_1'(1-\alpha)} = \frac{w_2'(\alpha)}{w_2'(1-\alpha)} \simeq \frac{\text{utility gain for 2}}{\text{utility loss for 2}}$$

has to be satisfied at each $\alpha \in [0,1]$.

With this interpretation in mind, we will now set up an exchange economy in which such bargaining weights can be traded. Formally, it is described by a tuple

$$\mathcal{E} = \mathcal{E}^{U^A,U^B} := ([0,1] \times [0,1], u_1, u_2, \omega_1, \omega_2)\,, \tag{8}$$

where $[0,1]^2$ reflects the commodity space for the two "commodities" *bargaining power in A and B*. As mentioned in the introduction of this section, utility functions are determined by adding utilities in the two bargaining problems, which means

$$u_i(\alpha, \beta) := w_i^A(\alpha) + w_i^B(\beta) \qquad (i = 1, 2).$$

Both agents are initially endowed with equal weights, i.e. $\omega_1 = \omega_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Note that the initial utility allocation is

$$\begin{aligned} (\ u_1\,(1/2, 1/2)\ ,\ u_2\,(1/2, 1/2)\ ) &= (\ \xi_1^A \left(\bar{b}^A/2\right) + \xi_1^B \left(\bar{b}^B/2\right)\ ,\ \xi_2^A \left(\bar{b}^A/2\right) + \xi_2^B \left(\bar{b}^B/2\right)\ ) \\ &= (\ \mu_1^A \left(U^A\right) + \mu_1^B \left(U^B\right)\ ,\ \mu_2^A \left(U^A\right) + \mu_2^B \left(U^B\right)\ )\,. \end{aligned}$$

Thus, initial utilities are given by the sum of PM solutions in the two underlying bargaining problems.

Lemma 2
For each agent i the utility function u_i is concave and strictly increasing. If both bargaining problems U^A, U^B are described by strictly concave functions C^A and C^B, then u_i is strictly concave.

Proof:
With the (strict) concavity of w_i^A and w_i^B $(i = 1, 2)$ one immediately obtains (strict) concavity of the utility functions u_1 and u_2, respectively. Use Lemma 1 to complete the proof. □

The superadditivity property will drive our results in two ways: First, as Perles & Maschler (1981a, Thm.5.1) show, for a fixed weight $\bar{\alpha}$ the mapping $\xi^{\cdot}(\bar{\alpha}) : \mathcal{U}_0^c \longrightarrow \mathbb{R}^2$ is a (non-symmetric) superadditive bargaining solution. Therefore, by Lemmas 1 and 2 we obtain concave utility functions and existence of Walrasian equilibria in $\mathcal{E}$.[7] By the First Welfare Theorem equilibrium allocations are Pareto efficient. In fact, the set of Pareto efficient allocations in $\mathcal{E}$ is mapped via (u_1, u_2) onto the set of Pareto efficient utility allocations in U. Second, superadditivity ensures that neither agent will lose when going to the solution of the aggregate problem. Therefore, the allocation of weights that corresponds to $\mu(U)$ is located in the Core of $\mathcal{E}$. With non-superadditive bargaining solutions this relation cannot be established. Note that any Walrasian equilibrium is also located in the Core. Our goal in the next section is to show that there is always a particular equilibrium in $\mathcal{E}$ that corresponds to the PM solution $\mu(U)$ of the aggregate bargaining problem. This means that the PM solution is achievable by "decentralized trading of weights".

4 Walrasian Equilibria and the PM Solution

Before we start equilibrium analysis in $\mathcal{E}$, we will need a couple of technical lemmas on the connection between standard traveling times and aggregation of bargaining problems. The following lemma is proved in Perles & Maschler (1981b, Cor.4.10).

[7] We thank an anonymous referee for pointing out that superadditivity is not a necessary condition to establish existence of equilibria.

Lemma 3
The function $\bar{b} : \mathcal{U}_0^c \longrightarrow \mathbb{R}$ that assigns to each bargaining problem its standard traveling time is additive on $\mathcal{U}_0^c$.

Lemma 4 is a well known result on efficient points in aggregate bargaining problems.

Lemma 4
A utility allocation $z \in U$ is Pareto efficient ($z \in \partial U$), if and only if there exist points $z^A \in \partial U^A$ and $z^B \in \partial U^B$ satisfying

$$z = z^A + z^B, \qquad NC_U(z) \cap NC_{U^A}(z^A) \cap NC_{U^B}(z^B) \neq \emptyset,$$

where $NC_U(z)$ denotes the set of supporting normal vectors at $z \in \partial U$.

For $z = (z_1, z_2) \in \partial U$ define $T^l_{U,z} := compH((U - z_l \cdot e^l) \cap \mathbb{R}^2)$ ($l = 1, 2$). We call $(T^1_{U,z}, 0) \in \mathcal{U}_0^c$ $((T^2_{U,z}, 0))$ the *truncated bargaining problem* of U in direction of the first (second) axis.

Lemmas 3 and 4 together yield a helpful connection between traveling times and efficient points.

Lemma 5
Suppose $z^A = (z_1^A, z_2^A) \in \partial U^A$ and $z^B = (z_1^B, z_2^B) \in \partial U^B$ are such that $NC_{U^A}(z^A) \cap NC_{U^B}(z^B) \neq \emptyset$ (which guarantees $z^A + z^B \in \partial U$).

1. For $l = 1, 2$ we have $T^l_{U^A,z^A} + T^l_{U^B,z^B} = T^l_{U,z^A+z^B}$.
2. Let s^A, s^B be determined by $\xi^A(s^A) = z^A, \xi^B(s^B) = z^B$. Then $\bar{b}(T^2_{U,z^A+z^B}) = s^A + s^B$ holds true.
3. Denote by s the corresponding traveling time for $z^A + z^B$, i.e. $\xi(s) = z^A + z^B = \xi^A(s^A) + \xi^B(s^B)$. Then we have $s = s^A + s^B$.

Proof:
To prove 1) use concavity of the functions C^A and C^B, which in particular means decreasing first derivatives. Then assertions 2) and 3) are a direct consequence of 1) and Lemma 3. $\square$

Now, let $z = (z_1, z_2) \in \partial U$ be Pareto efficient in U and $s \in [0, \bar{b}]$ with $\xi(s) := z$. From the construction of aggregate bargaining problems we know that z_2 can be expressed as the value of the following maximization problem:

$$\begin{aligned} z_2 &= \max\left\{C^A(t^A) + C^B(t^B) \mid t^A \in [0, \tau_1^A], t^B \in [0, \tau_1^B], \quad t^A + t^B = z_1\right\} \\ &= \max\left\{C^A(h^A(s^A)) + C^B(h^B(s^B)) \mid s^A \in [0, \bar{b}^A], \right. \\ &\qquad\qquad \left. s^B \in [0, \bar{b}^B], \quad \underbrace{h^A(s^A) + h^B(s^B) = h(s)}_{\xi_1^A(s^A)+\xi_1^B(s^B)=\xi_1(s)}\right\}. \end{aligned} \tag{9}$$

First order conditions (in the differentiable case) require $C^{A\prime}(h^A(s^A)) = C^{B\prime}(h^B(s^B))$. This means that necessarily we are in the situation of Lemma 5 and can therefore rewrite (9) to

$$z_2 = \max\left\{\xi_2^A(s^A) + \xi_2^B(s^B) \mid s^A \in [0, \bar{b}^A], s^B \in [0, \bar{b}^B], \quad s^A + s^B = s\right\}. \tag{10}$$

Analogously for z_1 we have

$$z_1 = \max\left\{\xi_1^A(s^A) + \xi_1^B(s^B) \mid s^A \in [0, \bar{b}^A], s^B \in [0, \bar{b}^B], \quad s^A + s^B = s\right\}. \tag{11}$$

In particular, the coordinates of the PM solution $\mu(U)$ are obtained from (10) and (11) with $s = \bar{b}/2$, i.e.

(12)

$$\mu_i(U) = \max\left\{\xi_i^A(s^A) + \xi_i^B(s^B) \mid s^A \in [0, \bar{b}^A], s^B \in [0, \bar{b}^B], \quad s^A + s^B = \frac{\bar{b}}{2}\right\} \quad (i = 1, 2).$$

Roughly speaking, the PM solution is obtained by efficiently splitting a total traveling time of $\bar{b}/2$ in traveling times s^A and s^B in A and B.

Coming back to the exchange economy Lemma 4 has the following direct implication.

Lemma 6

Let $((\alpha, \beta); (1-\alpha, 1-\beta))$ be an efficient allocation in $\mathcal{E}$. Then $\xi^A(\alpha\,\bar{b}^A) = \big(w_1^A(\alpha), w_2^A(1-\alpha)\big)$, which implies $h^A(\alpha\,\bar{b}^A) = w_1^A(\alpha)$ (analogously in situation B). Then the two derived utility allocations have a common normal vector, i.e. $NC_{U^A}(\xi^A(\alpha\,\bar{b}^A)) \cap NC_{U^B}(\xi^B(\beta\,\bar{b}^B)) \neq \emptyset$. Thus, in the differentiable case (and $0 < \alpha, \beta < 1$) the equation $C^{A\prime}(h^A(\alpha\,\bar{b}^A)) = C^{B\prime}(h^B(\beta\,\bar{b}^B))$ holds.

Proof:

Suppose to the contrary that α', β' are such that the referring utility allocations in U^A and

U^B do not have a common normal vector, i.e. $NC_{U^A}(\xi^A(\alpha' \bar{b}^A)) \cap NC_{U^B}(\xi^B(\beta' \bar{b}^B)) = \emptyset$. By Lemma 4 this means that the sum $\xi^A(\alpha' \bar{b}^A)) + \xi^B(\beta' \bar{b})$ is not located in ∂U, i.e. it is not efficient. Hence, there exists $z \in \partial U$ that dominates this sum. Again, by use of Lemma 4 there exist $z^A \in \partial U^A$ and $z^B \in \partial U^B$ with $NC_{U^A}(z^A) \cap NC_{U^B}(z^B) \neq \emptyset$ and $z^A + z^B = z$. Let α, β now be defined to satisfy $\xi^A(\alpha \bar{b}^A) = z^A$ and $\xi^B(\beta \bar{b}^B) = z^B$. Then $(u_1(\alpha,\beta), u_2(1-\alpha, 1-\beta)) = z^A + z^B \geq \xi^A(\alpha' \bar{b}^A)) + \xi^B(\beta' \bar{b}) = (u_1(\alpha', \beta'), u_2(1-\alpha', 1-\beta'))$ shows that $((\alpha', \beta'); (1-\alpha', 1-\beta'))$ is not efficient and the lemma is proved. □

Next, we address the question how equilibrium prices in $\mathcal{E}$ look like? For this, we look at agent 1's utility maximization problem. Suppose u_1, u_2 are differentiable. Note that for $\alpha \in [0,1]$ we have

$$w_1^{A'}(\alpha) = h^{A'}(\alpha \cdot \bar{b}^A) \cdot \bar{b}^A = \frac{\bar{b}^A}{\sqrt{(-C^{A'}(h^A(\alpha \cdot \bar{b}^A)))}}.$$

Furthermore, we know that in an equilibrium $((\bar{\alpha}, \bar{\beta}; 1-\bar{\alpha}, 1-\bar{\beta}), \bar{p}_1, \bar{p}_2)$ we have that the allocation is efficient and therefore $(u_1(\bar{\alpha}, \bar{\beta}), u_2(1-\bar{\alpha}, 1-\bar{\beta}))$ is located in ∂U.

In the differentiable case[8] we can achieve a result on equilibrium prices.

Theorem 1
Let $\mathcal{E}$ be an exchange economy as in (8) with differentiable utility functions u_1, u_2. Then there exists a Walrasian equilibrium with equilibrium prices $(\bar{p}_1, \bar{p}_2)$ that satisfy $\bar{p}_1/\bar{p}_2 = \bar{b}^A/\bar{b}^B$.

Proof:
We assume differentiability of C^A and C^B. For prices p_1 (for a unit of "power" in A) and p_2 we have the familiar first order conditions

$$\frac{\bar{p}_1}{\bar{p}_2} = \frac{\frac{\partial u_1}{\partial \alpha}(\bar{\alpha}, \bar{\beta})}{\frac{\partial u_1}{\partial \beta}(\bar{\alpha}, \bar{\beta})} = \frac{w^{A'}(\bar{\alpha})}{w^{B'}(\bar{\beta})} = \frac{\frac{\bar{b}^A}{\sqrt{(-C^{A'}(h^A(\alpha \cdot \bar{b}^A)))}}}{\frac{\bar{b}^B}{\sqrt{(-C^{B'}(h^B(\beta \cdot \bar{b}^B)))}}} = \frac{\bar{b}^A}{\bar{b}^B} \tag{13}$$

$$\bar{p}_1 \cdot \alpha + \bar{p}_2 \cdot \beta = \frac{1}{2} \cdot (\bar{p}_1 + \bar{p}_2) \tag{14}$$

(the last equation in (13) holds due to Lemma 6). □

[8] In the non-differentiable case the assertions have to be properly adjusted.

An inspection of agent 1's demand in an equilibrium for the two commodities with equilibrium prices $(\bar{p}_1, \bar{p}_2) = (\bar{b}^A, \bar{b}^B)$ now yields

$$
\begin{aligned}
\max_{\alpha,\beta}\left\{u_1(\alpha,\beta)\,|\,\bar{p}_1\,\alpha+\bar{p}_2\,\beta=\frac{\bar{p}_1+\bar{p}_2}{2}\right\} &= \max_{\alpha,\beta}\left\{w_1^A(\alpha)+w_1^B(\beta)\,|\,\bar{b}^A\,\alpha+\bar{b}^B\,\beta=\frac{\bar{b}^A+\bar{b}^B}{2}\right\}\\
&= \max_{\alpha,\beta}\left\{h^A(\alpha\cdot\bar{b^A})+h^B(\beta\cdot\bar{b}^B)\,|\,\bar{b}^A\,\alpha+\bar{b}^B\,\beta=\frac{\bar{b}}{2}\right\}\\
&= \max_{s^A,s^B}\left\{h^A(s^A)+h^B(s^B)\,|\,s^A+s^B=\frac{\bar{b}}{2}\right\} \qquad (15)\\
&= \max_{s^A,s^B}\left\{\xi_1^A(s^A)+\xi_1^B(s^B)\,|\,s^A+s^B=\frac{\bar{b}}{2}\right\}
\end{aligned}
$$

This means in view of (12) that given equilibrium prices as above agent 1 has to solve exactly the same maximization problem that also generates his coordinate of the PM solution. With similar considerations one obtains the same result for agent 2.

This establishes the following theorem.

Theorem 2

Let $\mathcal{E}$ be an exchange economy as in (8). Then there is an equilibrium $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta}),\bar{p}_1,\bar{p}_2)$ with equilibrium prices $(\bar{p}_1,\bar{p}_2) = (\bar{b}^A,\bar{b}^B)$ and the utility allocation in equilibrium coincides with the Perles-Maschler solution of the aggregate bargaining problem, i.e.

$$u_1(\bar{\alpha},\bar{\beta}) = \mu_1(U), \qquad u_2(1-\bar{\alpha},1-\bar{\beta}) = \mu_2(U).$$

Theorem 2 guarantees that the PM solution is achieved in some equilibrium with equilibrium prices that reflect the different traveling times. But still, there may be a large set of equilibrium prices. We shift the uniqueness question to the end of this section.

Next, we will have a closer look at Pareto efficient allocations and supporting prices. Due to concavity and monotonicity of utility functions the Second Fundamental Welfare Theorem applies to our exchange economy and therefore any efficient allocation can be described as an equilibrium with transfers (e.g. of initial endowments). Lemma 6 gives a necessary condition for efficient allocations in $\mathcal{E}$. In Theorem 1 this was exploited to show that the traveling times $\bar{b}^A$ and $\bar{b}^B$ in fact determine equilibrium prices. A re-inspection of the proof reveals that traveling times also determine supporting prices for arbitrary efficient allocations.

Lemma 7
Let $U \in \mathcal{U}_0^c$ be a bargaining problem and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ be a normal vector at U in $(\bar{t}, C(\bar{t}))$, i.e. $\lambda \in NC_U(\bar{t}, C(\bar{t}))$. Let $\bar{s}$ be the corresponding traveling time, which means $\xi(\bar{s}) = (\bar{t}, C(\bar{t}))$. Set $\bar{\alpha} := \bar{s}/\bar{b}$ the corresponding weight (for agent 1). Then $\lambda' := \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)$ is a normal vector for $w_1(\cdot)$ at $\bar{\alpha}$. To be precise, we assert

$$\lambda\left(\bar{t}, C(\bar{t})\right) \geq \lambda\left(t, C(t)\right) \quad (t \in [0, \tau_1]) \quad \text{ implies } \quad \lambda'\left(\bar{\alpha}, w_1(\bar{\alpha})\right) \geq \lambda'\left(\alpha, w_1(\alpha)\right) \quad (\alpha \in [0,1]).$$

Proof:
Fix $\bar{t} \in [0, \tau_1]$. Since λ is a supporting normal vector at $(\bar{t}, C(\bar{t}))$ one immediately concludes that

$$(16) \qquad \lambda_2 \cdot (-C'_{\searrow}(\bar{t})) \geq \lambda_1 \geq \lambda_2 \cdot (-C'_{\nearrow}(\bar{t}))$$

holds, where $C'_{\searrow}(\bar{t})$ ($C'_{\nearrow}(\bar{t})$) denotes the left-hand (right-hand) first derivative of C at $\bar{t}$. Due to concavity of the function C, the left inequality is valid for all $r \geq \bar{t}$ instead of $\bar{t}$, whereas the right inequality is valid for all $r \leq \bar{t}$. Taking appropriate integrals over square roots in (16) yields

$$\begin{aligned} \sqrt{\lambda_2} \cdot \int_{\bar{t}}^{t} \sqrt{-C'(r)}\, dr \geq \int_{\bar{t}}^{t} \sqrt{\lambda_1}\, dr \qquad & (t \geq \bar{t}) \\ \int_{t}^{\bar{t}} \sqrt{\lambda_1}\, dr \geq \sqrt{\lambda_2} \cdot \int_{t}^{\bar{t}} \sqrt{-C'(r)}\, dr \qquad & (t \leq \bar{t}), \end{aligned}$$

which is translated to

$$(17) \qquad \sqrt{\lambda_1}(t - \bar{t}) \leq \sqrt{\lambda_2}(f(t) - f(\bar{t})) \qquad (t \in [0, \tau_1]).$$

Since the mapping h is a bijection from $[0, \bar{b}]$ onto $[0, \tau_1]$, inequality (17) can be rewritten as

$$\begin{aligned} \sqrt{\lambda_1}(h(s) - h(\bar{s})) \leq \sqrt{\lambda_2}\left(f(h(s)) - f(h(\bar{s}))\right) \qquad & (s \in [0, \bar{b}]) \\ \sqrt{\lambda_2}(s - \bar{s}) \geq \sqrt{\lambda_1}(h(s) - h(\bar{s})) \qquad & (s \in [0, \bar{b}]) \\ \sqrt{\lambda_2}(\alpha\bar{b} - \bar{\alpha}\bar{b}) \geq \sqrt{\lambda_1}(h(\alpha\bar{b}) - h(\bar{\alpha}\bar{b})) \qquad & (\alpha \in [0,1]) \\ \bar{b}\sqrt{\lambda_2}(\alpha - \bar{\alpha}) \geq \sqrt{\lambda_1}(w_1(\alpha) - w_1(\bar{\alpha})) \qquad & (\alpha \in [0,1]) \\ \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\bar{\alpha}, w_1(\bar{\alpha})) \geq \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\alpha, w_1(\alpha)) \qquad & (\alpha \in [0,1]) \end{aligned}$$

The last inequality shows that $\lambda' = \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)$ is in fact a supporting normal vector for w_1 at $\bar{\alpha}$ and the lemma is proved. $\square$

Lemma 7 now enables us to derive supporting prices at an efficient allocation $((\bar\alpha,\bar\beta);(1-\bar\alpha,1-\bar\beta)$ in the Edgeworth box. By Lemma 6 we know that the two corresponding utility allocations $\xi^A(\bar\alpha\,\bar b^A)\in U^A$ and $\xi^B(\bar\beta\,\bar b^B)\in U^B$ have a common normal vector, say λ. From Lemma 7 we obtain the inequalities

$$(18)\quad \left(-\bar b^A\sqrt{\lambda_2},\sqrt{\lambda_1}\right)\left(\bar\alpha,w_1^A(\bar\alpha)\right)\geq\left(-\bar b^A\sqrt{\lambda_2},\sqrt{\lambda_1}\right)\left(\alpha,w_1^A(\alpha)\right)\qquad(\alpha\in[0,1])$$

$$(19)\quad \left(-\bar b^B\sqrt{\lambda_2},\sqrt{\lambda_1}\right)\left(\bar\beta,w_1^B(\bar\beta)\right)\geq\left(-\bar b^B\sqrt{\lambda_2},\sqrt{\lambda_1}\right)\left(\beta,w_1^B(\beta)\right)\qquad(\beta\in[0,1]).$$

Adding up (18) and (19) we get

$$\left(-\bar b^A\sqrt{\lambda_2},-\bar b^B\sqrt{\lambda_2},\sqrt{\lambda_1}\right)\left(\bar\alpha,\bar\beta,u_1(\bar\alpha,\bar\beta)\right)\geq\left(-\bar b^A\sqrt{\lambda_2},-\bar b^B\sqrt{\lambda_2},\sqrt{\lambda_1}\right)(\alpha,\beta,u_1(\alpha,\beta))$$

$$(20)\quad \left(\bar b^A\sqrt{\lambda_2},\bar b^B\sqrt{\lambda_2}\right)\left((\alpha,\beta)-\left(\bar\alpha,\bar\beta\right)\right)\geq\sqrt{\lambda_1}\left(u_1(\alpha,\beta)-u_1(\bar\alpha,\bar\beta)\right)\quad(\alpha,\beta\in[0,1]).$$

Inequality (20) now gives us the desired implication. Whenever agent 1 thinks the bundle (α,β) is at least as good as the "efficient bundle" $(\bar\alpha,\bar\beta)$, then the right hand side in (20) is not negative. This implies that the left hand side has to be non-negative and therefore the value of $(\bar\alpha,\bar\beta)$ under prices $(\bar b^A,\bar b^B)$ does not exceed the value of (α,β).

For agent 2 we get the analogous condition to (18) and (19) by interchanging λ_1 and λ_2. Thus the analogous inequality to (20) reads as

$$(21)\quad \left(\bar b^A\sqrt{\lambda_1},\bar b^B\sqrt{\lambda_1}\right)\left((\alpha,\beta)-\left(\bar\alpha,\bar\beta\right)\right)\geq\sqrt{\lambda_2}\left(u_2(\alpha,\beta)-u_2(\bar\alpha,\bar\beta)\right)\quad(\alpha,\beta\in[0,1]).$$

This establishes the following theorem.

Theorem 3
Let $((\bar\alpha,\bar\beta);(1-\bar\alpha,1-\bar\beta))$ be an efficient allocation in $\mathcal{E}$. Denote by K^A (K^B) the set of normal vectors supporting the respective utility allocations in U^A and U^B, i.e. $K^A:=NC_{U^A}(\xi^A(\bar\alpha\,\bar b^A))$ ($K^B:=NC_{U^B}(\xi^B(\bar\beta\,\bar b^B))$). Then the following statements hold:

1. The set of price vectors supporting u_1 at $(\bar\alpha,\bar\beta)$ is given by

   $$S_{u_1}(\bar\alpha,\bar\beta):=\left\{p=(p_1,p_2)\in\mathbb{R}^2_+\,|\,p=\left(\sqrt{\eta_2\,\rho_1}\,\bar b^A,\sqrt{\eta_1\,\rho_2}\,\bar b^B\right),\eta\in K^A,\rho\in K^B\right\}$$

   Analogously, the set of price vectors supporting u_2 at $(1-\bar\alpha,1-\bar\beta)$ is given by

   $$S_{u_2}(1-\bar\alpha,1-\bar\beta):=\left\{p=(p_1,p_2)\in\mathbb{R}^2_+\,|\,p=\left(\sqrt{\eta_1\,\rho_2}\,\bar b^A,\sqrt{\eta_2\,\rho_1}\,\bar b^B\right),\eta\in K^A,\rho\in K^B\right\}.$$

   Then the set of price vectors supporting the efficient allocation $((\bar\alpha,\bar\beta);(1-\bar\alpha,1-\bar\beta))$ is given by the intersection $S_{u_1}(\bar\alpha,\bar\beta)\cap S_{u_2}(1-\bar\alpha,1-\bar\beta)$.

2. In particular, the price system $(\bar{b}^A, \bar{b}^B)$ is a supporting price system for any efficient allocation in $\mathcal{E}$.

Proof:
In order to determine subgradients of u_1, consider inequality (18) with $\eta \in K^A$ instead of λ and (19) with $\rho \in K^B$ instead of λ. Multiplying the first inequality with $\sqrt{\rho_1}$ and the second with $\sqrt{\eta_1}$ yields

$$\left(-\bar{b}^A \sqrt{\rho_2\,\eta_1}, \sqrt{\rho_1\,\eta_1}\right)\left(\bar{\alpha}, w_1^A(\bar{\alpha})\right) \geq \left(-\bar{b}^A \sqrt{\rho_2\,\eta_1}, \sqrt{\rho_1\,\eta_1}\right)\left(\alpha, w_1^A(\alpha)\right) \qquad (\alpha \in [0,1])$$
$$\left(-\bar{b}^B \sqrt{\eta_2\,\rho_1}, \sqrt{\eta_1\,\rho_1}\right)\left(\bar{\beta}, w_1^B(\bar{\beta})\right) \geq \left(-\bar{b}^B \sqrt{\eta_2\,\rho_1}, \sqrt{\eta_1\,\rho_1}\right)\left(\beta, w_1^B(\beta)\right) \qquad (\beta \in [0,1]).$$

Summation now yields

$$\left(\bar{b}^A \sqrt{\rho_2\,\eta_1}, \bar{b}^B \sqrt{\eta_2\,\rho_1}\right)\left((\alpha,\beta) - \left(\bar{\alpha},\bar{\beta}\right)\right) \geq \sqrt{\eta_1\,\rho_1}\left(u_1(\alpha,\beta) - u_1(\bar{\alpha},\bar{\beta})\right) \quad (\alpha, \beta \in [0,1]).$$

This shows the support property for u_1 at $(\bar{\alpha},\bar{\beta})$. With analogous arguments and use of (21) we get the assertion for u_2.
The second part follows directly by taking $\rho = \eta \in K^A \cap K^B$. Then from part 1) the vector $\sqrt{\rho_1\,\rho_2}\left(\bar{b}^A, \bar{b}^B\right)$ (and hence the vector $\left(\bar{b}^A, \bar{b}^B\right)$) is located in $S_{u_1}(\bar{\alpha},\bar{\beta}) \cap S_{u_2}(1-\bar{\alpha}, 1-\bar{\beta})$. $\square$

Theorem 3 offers an extension of our results. It says that independent of the initial allocation, the price vector $(\bar{b}^A, \bar{b}^B)$ is always an equilibrium price vector. Moreover, as with symmetric endowments, both agents are better off in equilibrium, compared to the initial endowment. But now, all the desired properties (efficiency and superadditivity) are fulfilled.

We close the section with two sufficient conditions for uniqueness of equilibrium.

Corollary 1
Let $\mathcal{E}$ be an exchange economy as in (8). Assume that the functions C^A and C^B are differentiable. Let $((\bar{\alpha},\bar{\beta}); (1-\bar{\alpha}, 1-\bar{\beta}))$ be an efficient allocation with $0 < \bar{\alpha}, \bar{\beta} < 1$. Then $\left(\bar{b}^A, \bar{b}^B\right)$ is (up to normalization) the unique price vector supporting this allocation.

Proof:
Differentiability, efficiency and the non-boundary assumption together imply
$NC_{U^A}(\xi^A(\bar{\alpha}\,\bar{b}^A)) = NC_{U^B}(\xi^B(\bar{\alpha}\,\bar{b}^B)) =: \{q\,\lambda \mid q \in \mathbb{R}_{++}\}$. From Theorem 3 it follows $S_{u_1}(\bar{\alpha},\bar{\beta}) = S_{u_2}(1-\bar{\alpha}, 1-\bar{\beta}) = \left\{r\,(\bar{b}^A, \bar{b}^B) \mid r \in \mathbb{R}_{++}\right\}$, which proves the corollary. $\square$

Corollary 2
Let $\mathcal{E}$ be an exchange economy as in (8). Assume that the functions C^A and C^B are strictly concave, differentiable and satisfy

$$(22)\quad \lim_{t\searrow 0} C^{A\prime}(t) = 0 \qquad \lim_{t\nearrow \tau_1^A} C^{A\prime}(t) = \infty \qquad \lim_{t\searrow 0} C^{B\prime}(t) = 0 \qquad \lim_{t\nearrow \tau_1^B} C^{B\prime}(t) = \infty.$$

Then equilibrium prices are (up to normalization) uniquely determined by $\bar{p}_1/\bar{p}_2 = \bar{b}^A/\bar{b}^B$. Moreover, if C^A and C^B are strictly concave, then there exists exactly one equilibrium in $\mathcal{E}$.

Proof:
Condition (22) guarantees that the only efficient allocations, in which at least one of the weights is zero are those with either $(\bar{\alpha}, \bar{\beta}) = (0,0)$ or $(\bar{\alpha}, \bar{\beta}) = (1,1)$. But neither of these allocations can form an equilibrium. Thus, we are in the situation of Corollary 1 and therefore all efficient allocations are supported by a unique price vector. This establishes uniqueness of equilibrium prices. In case that C^A and C^B are strictly concave functions we know by Lemma 2 that utility functions u_1 and u_2 are strictly concave and therefore each agent's demand correspondence is single-valued, which implies that in this case there is exactly one equilibrium allocation. □

5 Examples

Example 1 (Non-differentiable case)
Consider the following setup:

$$C^A : [0,9] \longrightarrow \mathbb{R}, \quad C^A(t) := \begin{cases} 3 - \frac{1}{8}\,t & , \quad 0 \le t \le 8 \\ 18 - 2\,t & , \quad 8 < t \le 9 \end{cases} \qquad C^B : [0,2] \longrightarrow \mathbb{R}, \quad C^B(t) := 2 - t.$$

The bargaining problems are defined by

$$\begin{aligned} U^A &:= \mathit{compH}\left(\{z \in [0,9] \times [0,3] \,|\, z_2 \le C^A(z_1)\}\right) \\ U^B &:= \mathit{compH}\left(\{z \in [0,2] \times [0,2] \,|\, z_2 \le C^B(z_1)\}\right). \end{aligned}$$

Figure 1 illustrates the two bargaining problems and the aggregated one.

Standard traveling times are given by

$$\bar{b}^A = \int_0^9 \sqrt{-C^{A\prime}(s)}\, ds = 3\sqrt{2} \qquad\qquad \bar{b}^B = \int_0^2 \sqrt{-C^{B\prime}(s)}\, ds = 2.$$

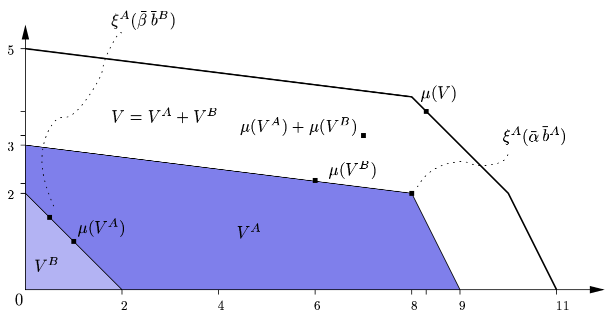

Figure 1: Bargaining Problems in Example 1

Straightforward computations reveal that standard parametrizations are given by

$$h^A(s) = \begin{cases} \sqrt{8}\, s & , \quad 0 \le s \le \sqrt{8} \\ \frac{1}{\sqrt{2}}\, s + 6 & , \quad \sqrt{8} < s \le 3\sqrt{2} \end{cases} \qquad \xi^A(s) = \big(h^A(s), C^A(h^A(s))\big)$$

$$h^B(s) = s, \qquad \xi^B(s) = (s, 2 - s)\,,$$

from which we can easily compute the PM solutions of U^A and U^B. We simply evaluate $\mu(U^A) = \xi^A(\frac{3}{2}\sqrt{2}) = (6, \frac{9}{4})$ and $\mu(U^B) = \xi^B(1) = (1, 1)$. As one immediately checks, the sum $\mu(U^A) + \mu(U^B)$ is not efficient in U. The PM solution of U is $\mu(U) = \big(9 - \frac{1}{2}\sqrt{2}, 3 + \frac{1}{2}\sqrt{2}\big)$ (see Figure 1).

From standard parametrizations we obtain the weight functions w^A, w^B

$$w_1^A(\alpha) = \begin{cases} \sqrt{8}\,(\alpha\, \bar b^A) = 12\,\alpha & , \quad 0 \le \alpha \le \frac{2}{3} \\ \frac{1}{\sqrt{2}}\,(\alpha\, \bar b^A) + 6 = 3\,\alpha + 6 & , \quad \frac{2}{3} < \alpha \le 1 \end{cases} \qquad w_2^A(\alpha) = \begin{cases} 6\,\alpha & , \quad 0 \le \alpha \le \frac{1}{3} \\ \frac{3}{2}\,\alpha + \frac{3}{2} & , \quad \frac{1}{3} < \alpha \le 1 \end{cases}$$

$$w_1^B(\beta) = 2\,\beta, \qquad w_2^B(\beta) = 2\,\beta.$$

which determines utilities as

$$u_1(\alpha, \beta) = \begin{cases} 12\,\alpha + 2\,\beta & , \quad 0 \le \alpha \le \frac{2}{3} \\ 3\,\alpha + 6 + 2\,\beta & , \quad \frac{2}{3} < \alpha \le 1 \end{cases} \qquad u_2(\alpha, \beta) = \begin{cases} 6\,\alpha + 2\,\beta & , \quad 0 \le \alpha \le \frac{1}{3} \\ \frac{3}{2}\,\alpha + \frac{3}{2} + 2\,\beta & , \quad \frac{1}{3} < \alpha \le 1. \end{cases}$$

Thus, the exchange economy $\mathcal{E}$ (cf. (8)) is completely defined. Figure 2 illustrates utility functions in the corresponding Edgeworth box. The solid lines represent agent 1's indifference curves, whereas dashed lines describe agent 2's indifference curves. The shaded area represents all individually rational allocations. As one can immediately see, there are

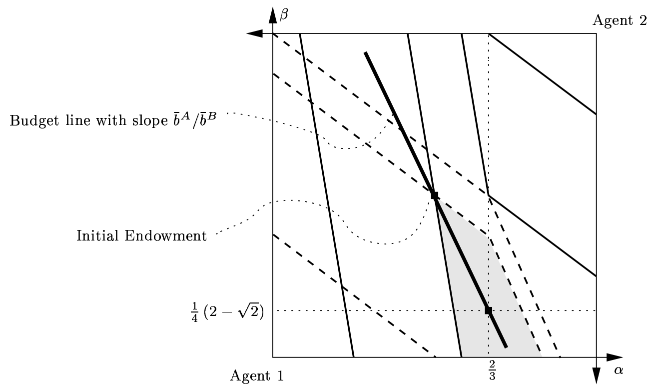

Figure 2: Edgeworth Box for Example 1

multiple equilibria in this exchange economy. If we computed the specific one with prices $(\bar{p}_1, \bar{p}_2) = (3\sqrt{2}, 2)$, we get a unique equilibrium allocation, which is $(\bar{\alpha}, \bar{\beta}; 1-\bar{\alpha}, 1-\bar{\beta})$ with $\bar{\alpha} = \frac{2}{3}$ and $\bar{\beta} = \frac{1}{4}(2-\sqrt{2})$ (cf. Figure 2). The utility allocation in this equilibrium is $(u_1(\bar{\alpha}, \bar{\beta}), u_2(1-\bar{\alpha}, 1-\bar{\beta}))) = \left(9 - \frac{1}{2}\sqrt{2}, 3 + \frac{1}{2}\sqrt{2}\right)$, which is indeed the PM solution of the aggregate bargaining problem U. $\square$

Example 2

We now consider an example with strictly concave functions C^A and C^B. The are given by

$$C^A : [0,2] \longrightarrow \mathbb{R}, \quad C^A(t) := 4 - t^2, \qquad C^B : [0, \ln 3] \longrightarrow \mathbb{R}, \quad C^B(t) := \frac{9}{2} - \frac{1}{2}\, e^{2\,t}.$$

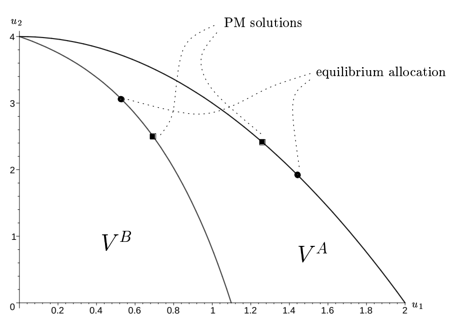

Figure 3: Bargaining Problems in Example 3

Again, by the same computations as in the examples above, we get

$$\bar{b}^A = \frac{8}{3}, \qquad \bar{b}^B = 2,$$

$$\xi^A(s) = \left(\frac{1}{2}\,(3\,s)^{\frac{2}{3}}; 4 - \frac{1}{4}\,(3\,s)^{\frac{4}{3}}\right), \qquad \xi^B(s) = \left(\ln(s+1); \frac{9}{2} - \frac{1}{2}\,(s+1)^2\right),$$

$$w^A(\alpha) = \left(2\,\alpha^{\frac{2}{3}}; 4 - 4\,(1-\alpha)^{\frac{4}{3}}\right), \qquad w^B(\beta) = \left(\ln(2\,\beta+1); \frac{9}{2} - \frac{1}{2}(3-2\,\beta)^2\right)$$

$$u_1(\alpha,\beta) = 2\,\alpha^{\frac{2}{3}} + \ln(2\,\beta+1) \qquad u_2(\alpha,\beta) = 4 - 4\,(1-\alpha)^{\frac{4}{3}} + \frac{9}{2} - \frac{1}{2}(3-2\,\beta)^2.$$

Figure 4 shows indifference curves for the two agents and the unique equilibrium alloca-

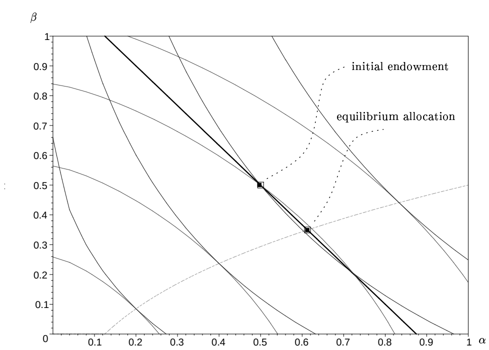

Figure 4: Edgeworth Box for Example 2

tion. Indeed, the equilibrium are according to Theorem 1 given by the standard traveling times. The equilibrium $((\bar{\alpha}, \bar{\beta}; 1-\bar{\alpha}, 1-\bar{\beta}), \bar{p}_1, \bar{p}_2)$ is given by

$$\bar{\alpha} \approx 0.3494, \qquad \bar{\beta} \approx 0.6129, \qquad \bar{p_1} = \frac{8}{3}, \qquad \bar{p_2} = 2. \; \square$$

6 Concluding Remarks

One may as well think of other bargaining solutions and their asymmetric versions to get a similar construction for bargaining power. Yet, it turns out that even if the weight functions $w^{\cdot}$ have the desired properties, the final solution (of the aggregate), loosely speaking, does not have to be in the Core of the exchange economy. Hence, it cannot be established by an equilibrium.

The most frequently used class of asymmetric bargaining solutions is the class of Nash solutions. With an (asymmetric) Nash solution, weight functions may fail to be strictly increasing or to be concave. As a result, preferences in the Edgeworth box may no longer be convex and hence existence of an equilibrium is not guaranteed. However, in view of the present model, the superadditivity axiom offers the most "natural" property: it provides an incentive for both agents to consider all bargaining problems simultaneously.

Some readers may feel uncomfortable with a seemingly conflicting mixture of cardinal and ordinal solution concepts. Indeed, as soon as we enter the Edgeworth box and apply the Walrasian equilibrium concept, we are no longer in a cardinal context. Yet, we view this way as a tool to come up with a certain allocation of bargaining power. And exactly this allocation is meant to "execute" the utilities, i.e. to determine the solution in the cardinal context. Note that agents' preferences in the Edgeworth box are not touched by the right transformations of the two bargaining problems. If we apply the same linear transformation to both bargaining problems, then the agents' utility functions will be linearly transformed and hence preferences will be preserved.[9]

The work in the paper can be extended in a couple of directions. First, the class of bargaining problems under consideration can be extended from $\mathcal{U}_0^c$ to $\mathcal{U}^c$ without substantial change of the results. This is as unproblematic as allowing boundaries of utility possibility sets to contain line segments that are parallel to some axis. Finally, there is nothing special with the fact that we consider two bargaining problems. With analogous arguments as used in the paper, one can consider the model with finitely many bargaining situations. Also note from Theorem 3 it follows that the results can easily be modified for asymmetric versions of the PM solution as well. If one starts with an identical but not equal initial endowment of weights in U^A and U^B, one obtains an equilibrium that corresponds to the asymmetric PM solution w.r.t. the starting weights. The straightforward details are left to the reader.

Since there is no superadditive bargaining solution for more than two persons (see Perles (1982)), we cannot hope for a straightforward extension of our model to the n-person case. Whether or not the extension of the PM solution to n-person bargaining problems can be used to define a notion of bargaining power is an open problem. But the lack of superadditivity may be an insurmountable obstacle for the process of finding an agreement.

[9] Application of different linear transformations to the bargaining problems should not be allowed, because this would violate our assumption that an agent's overall utility is the sum of utilities he gets in the two bargaining problems.

References

CALVO, E. & E. GUTHERREZ (1994): "Extension of the Perles-Maschler Solution to N-Person Bargaining Games", *International Journal of Game Theory*, 23, 325–364.

NASH, J. F. (1950): "The Bargaining Problem", *Econometrica*, 18, 155–162.

PERLES, M. (1982): "Non-Existence of Super-Additive Solutions for 3-Person Games", *International Journal of Game Theory*, 11(3/4), 151–161.

PERLES, M. & M. MASCHLER (1981a): "The Present Status of the Superadditive Solution", in *Essays in Game Theory and Mathematical Economics*, ed. by R. Aumann et al., pp. 103–110. Bibliographisches Institut, Mannheim.

——— (1981b): "The Super-Additive Solution for the Nash Bargaining Game", *International Journal of Game Theory*, 10(3/4), 163–193.

PETERS, H. J. M. (1992): *Axiomatic bargaining game theory*, Theory and decision library : Ser. C ; 9. Kluwer, Dordrecht [u.a.].

ROSENMÜLLER, J. (2000): *Game Theory: Stochastics, Information, Strategies and Cooperation*, vol. 25 of *Theory and Decision Library (Series C)*. Kluwer Academic Publishers, Boston / Dordrecht / London.